c) Montrer que pour tout $n \geq 1$, $U_{n+1} + U_n = \frac{1}{2n+1}$. En déduire $\lim\limits_{n \to +\infty} U_n$.

$$U_{n+1} + U_n = \int_0^{\frac{\pi}{4}} f^{n+1}(x)\,dx + \int_0^{\frac{\pi}{4}} f^{n}(x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(f^{n+1}(x) + f^{n}(x) \right) dx$$

$$= \int_0^{\frac{\pi}{4}} f^{n}(x) \left(f(x) + 1 \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(1 + \tan^2(x) \right) \cdot \underbrace{\tan^{2n}(x)}_{U^{2n}} dx$$

U'

$$= \left[\frac{\tan^{2n+1}(x)}{2n+1} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\tan^{2n+1}\left(\frac{\pi}{4}\right)}{2n+1} - 0$$

$$= \frac{1}{2n+1}$$

$\forall n \geq 1$ $\quad U_{n+1} + U_n = \frac{1}{2n+1}$